L'Hospital's Rule II $\frac{\infty}{\infty}$

Let I be an interval and f, g are differentiable functions on I such that $g'(x) \neq 0$. Suppose $\lim_{x \to a} g(x) = \pm\infty$.

If $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$ where $L \in \mathbb{R} \cup \{\pm\infty\}$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = L$.

Pf:

- Suppose $L \in \mathbb{R}$. Assume $L > 0$. Fix $\varepsilon > 0$. Assume $\varepsilon < 1$,
  $\varepsilon < L$.

  Note that
  $$\frac{f(x)}{g(x)} = \frac{f(x) - f(y)}{g(x)} + \frac{f(y)}{g(x)}$$
  $$= \frac{g(x) - g(y)}{g(x)} \frac{f(x) - f(y)}{g(x) - g(y)} + \frac{f(y)}{g(y)}$$
  $$= \left(1 - \frac{g(y)}{g(x)}\right) \frac{f(x) - f(y)}{g(x) - g(y)} + \frac{f(y)}{g(x)}$$

  Since $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$, there exists $\delta_1 > 0$ such that for any $x \in I$ satisfying $a < x < a + \delta_1$,
  $$\left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon.$$

  Now we fix $y \in (a, a + \delta_1)$.

For any $x \in (a, y)$, by Rolle's Theorem,

$g(x) \neq g(y)$ since $g' \neq 0$ on I.

By Cauchy's MVT, there exists some $c \in (x, y)$ such that

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(c)}{g'(c)} \in (L-\varepsilon, L+\varepsilon).$$

Since $\lim\limits_{x\to a} g(x) = \infty$, then $\lim\limits_{x\to a} \frac{g(y)}{g(x)} = \lim\limits_{x\to a} \frac{f(y)}{g(x)} = 0$.

Then there exists $\delta_2 > 0$ such that for any $x \in I$ satisfying $a < x < a+\delta_2$,

$$\left|\frac{g(y)}{g(x)}\right|, \left|\frac{f(y)}{g(x)}\right| < \varepsilon.$$

Take $\delta := \min\{y-x, \delta_2\}$.

Then for any $x \in I$ satisfying $a < x < a+\delta$,

$$(1-\varepsilon)(L-\varepsilon) - \varepsilon < \left(1 - \frac{g(y)}{g(x)}\right) \frac{f(x)-f(y)}{g(x)-g(y)} + \frac{f(y)}{g(x)} < (1+\varepsilon)(L+\varepsilon) + \varepsilon$$

i.e. $$L-(L+2+\varepsilon)\varepsilon < \frac{f(x)}{g(x)} < L+(L+2+\varepsilon)\varepsilon$$

Now we have proved $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$ when $L>0$.

Similarly, we can prove $\lim_{x\to a^-} \frac{f(x)}{g(x)} = L$ and

the cases $L<0$ or $L=0$.

- Suppose $L=\infty$.

  Note that $\frac{f(x)}{g(x)} = \left(1-\frac{g(y)}{g(x)}\right)\frac{f(x)-f(y)}{g(x)-g(y)} + \frac{f(y)}{g(x)}$

  Fix $M>0$.

  Since $\lim_{x\to a}\frac{f'(x)}{g'(x)} = \infty$, there exists some

  $\delta_1>0$ such that $\frac{f'(x)}{g'(x)} > M$ for any $x\in(a, a+\delta_1)$.

  Fix $y\in(a, a+\delta_1)$.

  By Rolle's Theorem and Cauchy's MVT,

  for any $x\in(a,y)$, there exists $c\in(x,y)$ s.t.

  $$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(c)}{g'(c)} > M.$$

  Since $\lim_{x\to a} g(x) = \infty$, $\lim_{x\to a}\frac{f(y)}{g(x)} = \lim_{x\to a}\frac{g(y)}{g(x)} = 0$.

Then $\exists\ \delta_2 > 0$ such that for any

$a < x < a + \delta_2$, $\quad \frac{f(y)}{g(x)} > -\frac{M}{4}$, $\quad \frac{g(y)}{g(x)} < \frac{1}{2}$.

Take $\delta := \min\{y - x, \delta_2\}$.

For any $x \in (a, a+\delta)$,

$$\frac{f(x)}{g(x)} = \left(1 - \frac{g(y)}{g(x)}\right) \frac{f(x) - f(y)}{g(x) - g(y)} + \frac{f(y)}{g(x)}$$

$$> \frac{1}{2} M - \frac{M}{4}$$

$$= \frac{M}{4}$$

Therefore, $\lim\limits_{x \to a^+} \frac{f(x)}{g(x)} = \infty$.

Similarly, we can prove $\lim\limits_{x \to a^-} \frac{f(x)}{g(x)} = \infty$ and

the case $L = -\infty$.

Rmk: The argument still works when $a = \pm\infty$.

Examples.

- $\lim_{x\to\infty} \frac{\ln x}{x}$

Note that $\lim_{x\to\infty} x = \infty$, $\lim_{x\to\infty} \frac{(\ln x)'}{x'} = \lim_{x\to\infty} \frac{1/x}{1} = 0$.

By L'Hospital's Rule , $\lim_{x\to\infty} \frac{\ln x}{x} = 0$

- $\lim_{x\to\infty} e^{-x}x^2$

Let $f(x) = x^2$ and $g(x) = e^x$

Then $f'(x) = 2x$ and $g'(x) = e^x$.

$f''(x) = 2$ and $g''(x) = e^x$.

Thus $\lim_{x\to\infty} e^{-x}x^2 = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$.

- $\lim_{x\to 0} \frac{\ln \sin x}{\ln x}$

Let $f(x) = \ln \sin x$ and $g(x) = \ln x$

Then $f'(x) = \frac{\cos x}{\sin x}$ and $g'(x) = \frac{1}{x}$

Thus $\lim_{x\to 0} \frac{\ln \sin x}{\ln x} = \lim_{x\to 0} \frac{\cos x / \sin x}{1/x} = \lim_{x\to 0} \frac{x \cos x}{\sin x}$

$= \left(\lim_{x\to 0} \frac{x}{\sin x}\right)\left(\lim_{x\to 0} \cos x\right) = 1$.